A THEOREM CONCERNING THE INFINITE CARDINAL NUMBERS.

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§ 1. THE principal object of this paper is to prove rigorously that the cardinal number of the continuum is *greater than or equal to* the cardinal number of Cantor's second number class; in symbols, that

$$2^{\alpha_0} \geqq \alpha_1,$$

and, more generally, that

$$2^{\alpha_\beta} \geqq \alpha_{\beta+1},$$

where β is any ordinal, and α_β the β^{th} cardinal contained in the well ordered series of cardinals constructed by Cantor.* I do not claim any great degree of originality for this theorem, for, as I shall show in § 2, it follows at once from Cantor's theorem that

$$2^{\alpha} > \alpha,$$

where α is any cardinal (whether contained in the above mentioned well ordered series or not), and some general consideration of the nature of the cardinals contained in the series. But the theorem has not, as far as I know, ever been stated explicitly. Indeed, in so recent a work as Mr. Russell's *Principles of Mathematics*, it is expressly stated that 'it may be that 2^{α_0} is neither greater nor less than α_1 and α_2 and their successors.'† And it seems to me that the investigation of § 3, in which I construct a set of points of cardinal α_1 in the linear continuum, may be of some interest as throwing some light (though of course a very partial one) on one of the most fundamentally important and apparently hopeless questions in the whole range of pure mathematics.

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* Schoenflies, *Mengenlehre*, p. 49. I use α_0, α_1, ... for Cantor's Aleph-Zero, Aleph-Eins, ... Mr. Whitehead and Mr. Russell use a_0, a_1, ...

† *l. c.*, p. 323.

§ 2. It is, however (as I have already remarked), possible to prove that $2^{\alpha_0} \geqq \alpha_1$ without using the construction of § 3. For, although we are not entitled to assume that the cardinals of all aggregates occur in the series

$$\alpha_0, \alpha_1, \ldots, \alpha_\omega, \ldots, \alpha_\beta, \ldots,$$

it does seem to me clear that any cardinal must either occur in the series or be *greater than all* the terms of the series.

This can, I think, be shown by an extension of Cantor's argument proving that every infinite cardinal is greater than or equal to α_0. For, given any aggregate whose cardinal $> \alpha_0$, we can choose from it successively individuals

$$u_1, u_2, \ldots, u_\omega, \ldots, u_\beta, \ldots$$

corresponding to all the numbers of the first and second classes; if the process came to an end, the cardinal of the aggregate would be α_0. Its cardinal therefore $\geqq \alpha_1$; and if $> \alpha_1$, $\geqq \alpha_2$, and so on. And if $> \alpha_n$, for all finite values of n, it must be $\geqq \alpha_\omega$; for we can choose individuals from the aggregate corresponding to all the numbers of the first, second, third, ..., n^{th}, ... classes. And by a repetition of these two arguments, we can show that if there is no α_β equal to the cardinal of the aggregate, it must be at least equal to the cardinal of the aggregate of all α_β's, and so greater than any α_β.

These considerations must, I imagine, have been familiar to Cantor. But he confines himself to showing that there is no cardinal *between* α_0 and α_1, *i.e.* $> \alpha_0$, but $< \alpha_1$, and never explicitly rejects the possibility contemplated by Mr. Russell.

§ 3. I come now to the actual construction of a set of points of cardinal α_1. I am not aware that such a set has been constructed before, unless indeed $2^{\alpha_0} = \alpha_1$; for all known sets have α_0 or 2^{α_0} as their cardinal.

Starting from the sequence of integral numbers,

$$(1) \quad 1, 2, 3, 4, 5, \ldots$$

we form a new sequence,

$$(2) \quad 2, 3, 4, 5, 6, \ldots$$

by omitting the first term; and by continuing the process we form

$$(3) \quad 3, 4, 5, 6, 7, \ldots$$

$$(4) \quad 4, 5, 6, 7, 8, \ldots$$

$$(5) \quad 5, 6, 7, 8, 9, \ldots$$

$$\ldots\ldots\ldots\ldots$$

We now form a new sequence

$$(\omega) \quad 1, 3, 5, 7, 9, \ldots$$

by traversing the above infinite array of sequences diagonally. Then we form

$$(\omega+1) \quad 3, \ 5, \ 7, \ 9, 11, \ldots$$

$$(\omega+2) \quad 5, \ 7, \ 9, 11, 13, \ldots$$

$$(\omega+3) \quad 7, \ 9, 11, 13, 15, \ldots$$

$$(\omega+4) \quad 9, 11, 13, 15, 17, \ldots$$

$$\ldots\ldots\ldots\ldots$$

$$(\omega.2) \quad 1, \ 5, \ 9, 13, 17, \ldots$$

$$(\omega.2+1) \quad 5, \ 9, 13, 17, 21, \ldots$$

$$(\omega.2+2) \quad 9, 13, 17, 21, 25, \ldots$$

$$(\omega.2+3) \quad 13, 17, 21, 25, 29, \ldots$$

$$\ldots\ldots\ldots\ldots$$

$$(\omega.3) \quad 1, \ 9, 17, 25, 33, \ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Thus we form sequences corresponding to all the numbers

$$\omega.\mu + \nu,$$

where μ and ν are finite.*

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* The method employed is similar in principle to that used by Borel to form a non-enumerable sequence of 'croissances'; v. *Leçons*, 1, p. 114.

To form the sequence corresponding to ω^2 we take the array of sequences

$$\begin{array}{llllll}
(\omega) & 1, & 3, & 5, & 7, & 9, \ldots \\
(\omega.2) & 1, & 5, & 9, & 13, & 17, \ldots \\
(\omega.3) & 1, & 9, & 17, & 25, & 33, \ldots \\
(\omega.4) & 1, & 17, & 33, & 49, & 65, \ldots \\
(\omega.5) & 1, & 33, & 65, & 97, & 129, \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}$$

and traverse it diagonally; so that we obtain

$$(\omega^2) \quad 1, 5, 17, 49, 129, \ldots.$$

Generally, if

$$b_1, b_2, b_3, b_4, \ldots$$

corresponds to β,

$$b_2, b_3, b_4, b_5, \ldots$$

corresponds to $\beta+1$; while to obtain a set corresponding to a number γ which has no predecessor, we take the array of sequences corresponding to any ascending set of numbers

$$\beta_1, \beta_2, \ldots$$

whose limit is γ, and traverse it diagonally, as has been shown in the particular cases of the numbers

$$\omega, \omega.2, \ldots, \omega^2.$$

We might equally well have considered ω as the limit of

$$1, 3, 5, 7, \ldots,$$

or ω^2 as the limit of

$$\omega+1, \omega.2+2, \omega.3+3, \ldots;$$

in fact we have an infinite freedom of choice whenever we wish to define the sequence corresponding to any number which has no immediate predecessor. This freedom of choice is important, as we are able, by exercising it suitably, to make it clear that all the sets we obtain are distinct.

It is clear that as we can always define a sequence corresponding to $\beta+1$ if we know one corresponding to β, and a sequence corresponding to $\gamma = \lim. \beta_m$ if we know those corresponding to $\beta_1, \beta_2, \ldots, \beta_m, \ldots$, we can certainly find sequences

corresponding to all the numbers of the second class. Hence, if we prove that all the sequences are distinct, their aggregate will have $\aleph_1$ as its cardinal.

§4. I shall now establish this by proving that we can so construct our sequences

$$b_1, b_2, b_3, \ldots$$

that in every case $b_1 < b_2 < b_3 \ldots$, and that, if $b_1, b_2, b_3, \ldots$ and $b_1', b_2', b_3', \ldots$ correspond to β and β', and $\beta < \beta'$, there exists a number N such that

$$b_n' > b_n \quad (n \geqq N).$$

Let us assume that we have constructed sequences corresponding to all the numbers $< \gamma$ in such a way that they satisfy this condition. Then there are two cases to consider, that in which γ has an immediate predecessor γ', and that in which it has not.

In the first place let $\gamma = \gamma' + 1$. Then if $\beta < \gamma'$ there is a number N such that

$$a_n' > b_n \quad (n \geqq N).$$

But $$a_n = a'_{n+1} > a_n' > b_n \quad (n \geqq N).$$

Hence, if the construction is posible for all numbers $< \gamma$, it is possible for all numbers $\leqq \gamma$.

Next suppose that γ has no immediate predecessor, and that

$$\gamma = \lim. \beta_m \quad (\beta_1 < \beta_2 < \beta_3 \ldots),$$

then $$\gamma = \lim. (\beta_m + \nu_m),$$

where the ν's are any finite numbers. Now there is a number N_1 such that

$$b_{2,n} > b_{1,n}, \quad (n \geqq N_1),$$

$b_{m,n}$ being the n^{th} number in the sequence corresponding to β_m. *A fortiori*, if $\gamma_m = \beta_m + \nu_m$,

$$c_{2,n} = b_{2,n+\nu_2} > b_{2,n} > b_{1,n} \quad (n \geqq N_1).$$

But if we take $\nu_2 > b_{1,N_1-1}$

$$c_{2,n} = b_{2,n+\nu_2} \geqq n + \nu_2 > b_{1,N_1-1} > b_{1,n} \quad (n < N_1),$$

Hence $c_{2,n} > b_{1,n}$ for *all* values of n. Similarly we can choose ν_3 so that $\gamma_3 > \gamma_2$, and $c_{3,n} > c_{2,n}$ for all values of n; and so on generally.

For the sake of uniformity I write γ_1 for β_1, $c_{1,n}$ for $b_{1,n}$. Then we have a doubly infinite array

$$\begin{array}{l} c_{1,1},\ c_{1,2},\ c_{1,3},\ \ldots, \\ c_{2,1},\ c_{2,2},\ c_{2,3},\ \ldots, \\ c_{3,1},\ c_{3,2},\ c_{3,3},\ \ldots, \\ \ldots\ldots\ldots\ldots\ldots\ldots, \end{array}$$

and we define the sequence corresponding to γ by traversing it diagonally, so that

$$c_n = c_{n,n}.$$

If then $\beta < \gamma$, we can find m so that

$$\beta < \gamma_m.$$

Then there is a number K such that

$$c_{m,n} > b_n \quad (n \geqq K).$$

But if $n > m$,

$$c_n = c_{n,n} > c_{m,n}.$$

If therefore n is greater than the greater of m, K

$$c_n > b_n.$$

And therefore, if the construction is possible for all numbers $< \gamma$, it is possible for all numbers $\leqq \gamma$, whether γ has a predecessor or not. But it is evidently possible for small values of γ. Thus it is possible to carry out the construction so as to obtain $\aleph_1$ distinct sequences.

In the case of the comparatively early numbers of the second class it is generally evident that a γ which has no predecessor is most naturally regarded as the limit of one particular set $\beta_1, \beta_2, \ldots$. Thus it is natural to regard ω^ω as the limit of

$$1,\ \omega,\ \omega^2,\ \omega^3,\ \ldots,$$

ω^{ω^ω} as the limit of

$$\omega,\ \omega^\omega,\ \omega^{\omega^2},\ \omega^{\omega^3},\ \ldots,$$

and ϵ_1, the first of Cantor's ϵ-numbers as the limit of

$$\omega,\ \omega^\omega,\ \omega^{\omega^\omega},\ \ldots.$$

And it is unnecessary to introduce the numbers ν, it being evident that the sequences which we obtain are all distinct. Thus I find

$$\begin{array}{rl}(\omega^2+1) & 5,\ 17,\ 49,\ 129, \ldots\\ (\omega^2+\omega) & 1,\ 17,\ 129, \ldots\\ (\omega^2+\omega.2) & 1,\ 129, \ldots\\ (\omega^2.2) & 1,\ 17, \ldots\\ (\omega^3) & 1,\ 17, \ldots\\ (\omega^\omega) & 1,\ 5, \ldots\\ (\epsilon_1) & 1,\ 5, \ldots.\end{array}$$

It will be seen that (ω^ω) and (ϵ_1) have the same two first terms, and we should find with higher numbers sequences which, though distinct, had more than any fixed finite number of terms in common. This will be clear when we have correlated the sequences with points in the linear continuum. The simplest way of effecting this correlation is perhaps to correlate

$$b_1,\ b_2,\ b_3,\ \ldots$$

with the binary decimal in which the b_1^{th}, b_2^{th}, ... figures are 1's and the remaining figures 0's. As the set of points thus obtained is non-enumerable, it must contain α_1 of its own limit points. Thus there are α_1 sequences such that, however great be n, we can find other sequences agreeing with them in their first n terms.

I may remark that we can divide the numbers 1, 2, ... into two classes, those which do not appear in any of the sequences after a certain one, and those which appear in a non-enumerable infinity of sequences. The latter we may call *persistent* numbers. If μ_1, μ_2, ... are the non-persistent numbers, we can find numbers of the second class

$$\gamma_1,\ \gamma_2,\ \ldots,$$

such that μ_n does not appear in any sequences after the γ_n^{th}. If γ is the first number greater than any of γ_1, γ_2, ..., all the sequences after the γ^{th} are composed entirely of persistent numbers.

§ 5. A similar method may be employed to show that generally

$$2^{\alpha_\beta} \geqq \alpha_{\beta+1}.$$

For instance, if $\beta = 1$, we start with the complete sequence

$$(1) \quad 1, 2, 3, \ldots, \omega, \omega+1, \ldots, \omega.2, \ldots, \omega^2, \ldots$$

of the numbers of the first and second classes, and form successively the sequences

$$(2) \quad 2, 3, 4, \ldots, \omega+1, \omega+2, \ldots, \omega.2+1, \ldots, \omega^2+1, \ldots$$

$$(3) \quad 3, 4, 5, \ldots, \omega+2, \omega+3, \ldots, \omega.2+2, \ldots, \omega^2+2, \ldots$$

$$(4) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

by increasing each term by $1, 2, \ldots$. We may obtain the sequence corresponding to ω by considering (1) as made up of α_1 sequences of the type

$$1, 2, 3, \ldots,$$

an aggregate of sequences which is ordinally similar to (1). We then get (ω) from (1) by moving these sequences in the same way in which we moved the single numbers when we obtained (2) from (1). Thus we find

$$(\omega) \quad \omega, \omega+1, \omega+2, \ldots, \omega.2, \ldots, \omega^2+\omega, \ldots.$$

Similarly we find

$$(\omega.2) \quad \omega.2, \omega.2+1, \ldots, \omega.3, \ldots, \omega^2+\omega.2, \ldots.$$

To find the sequence corresponding to ω^2, we consider (1) as made up of α_1 sequences of the type ω^2; this aggregate of sequences is again ordinally similar to (1).

By following this method we can find sequences for all the numbers of the second class. Then, by traversing the whole array of α_1 sequences we obtain a sequence for Ω, the first number of the third class. And then there is no difficulty in finding sequences for all the numbers of the third class. And these sequences may be correlated with a part of the aggregate of all possible sequences of α_1 figures each of which is either 0 or 1, an aggregate whose cardinal is clearly 2^{α_1}.* But I shall not attempt for this (still less for the general case) an investigation similar to that of § 4, but shall content myself with the general line of proof indicated in § 2, and the detailed construction given in §§ 3, 4 for the most interesting case.

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* Mr. Whitehead worked out some of the most interesting properties of such aggregates in his lectures on the application of symbolic logic to the theory of aggregates, delivered during the winter of 1902-3.